APPROXIMATE DETERMINATION OF CONDUCTIVITY IN TWO-DIMENSIONAL (PLANE) FIGURES

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Justification and examples are provided for the utilization of a method to evaluate the conductivity in two-dimensional figures of complex shape. Two methods are proposed for the derivation of approximate conductivity formulas.

Given a body of infinite length and constant cross section, having a steady distribution of potential (with two isopotential surfaces bounding the body) and subject to the Laplace equation under boundary conditions of the 1-st kind, we solve the problem of the distribution of the potential in such a body simply by examining the two-dimensional figure which serves as the lateral cross section. The conductivity of the two-dimensional figure is understood to be a purely geometric quantity (the shape factor [1]), directly proportional to the flow of energy or matter per unit length of the corresponding body and inversely proportional to the difference in potentials at the boundaries and to the corresponding transport coefficient. For determinacy we will speak, in the following, of a temperature field and a heat flux Q through the solid body with a constant coefficient of thermal conductivity λ, so that the conductivity Π is defined from the equation $Q = \lambda\Pi\Delta t$. In the formulated statement of the problem, the purpose of the engineering calculation is generally the determination of conductivity. Here it is sufficient to find the conductivity of the two-dimensional figure which makes up a part of the lateral cross section of the body and is bounded by two isotherms and by two adiabatic curves, drawn with consideration of the conditions of symmetry for the temperature field.

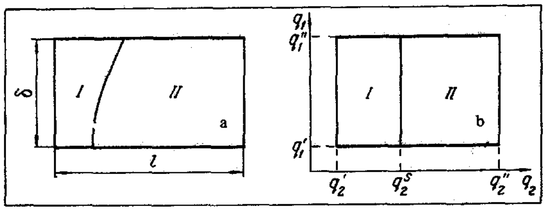

Fig. 1. Demonstration of the main statement of the division method (here and subsequently, the bold lines show boundary isotherms): a) figure division with adiabatic line; b) composite figure in coordinates q_1, q_2.

Exact formulas for the calculation of conductivity exist only for such comparatively simple figures as, for example, a rectangle and a sector of a concentric ring. For figures of greater complexity it becomes necessary to resort to physical modeling or the tedious calculation of the temperature field by a numerical method. Particularly productive, in our opinion, is the method of the approximate calculation of conductivity employed in [2–4] with respect to the calculation of insulation in electrical equipment. That method is based on the division of a complex figure by means of adiabatic and isothermal lines into simple parts; this is then followed by the determination of conductivity in the composite figure on the basis of well-known formulas. The clear advantage of this method lies in its simplicity and greater generality, relative to the numerical method and the method of modeling. The method makes it possible to establish an approximate formula for the conductivity of an entire class of similar figures, with changes only in their geometric parameters. Unfortunately, this method was employed in the cited references without any justification, nor without any reference to its approximate nature. An attempt is made here to provide a rigorous foundation for the method of division and for its subsequent development.

We will adopt a system of isotherms and adiabatic curves (streamlines) in a two-dimensional figure as an orthogonal curvilinear system of coordinates $q_1 = t$, $q_2 = \psi$, where t is the temperature and ψ is the stream function. We will write the familiar Laplace equation for this new coordinate system [5] as follows:

$$\frac{\partial}{\partial q_1}\left(\frac{H_2}{H_1}\frac{\partial t}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{H_1}{H_2}\frac{\partial t}{\partial q_2}\right) = 0. \qquad (1)$$

Here H_1 and H_2 are functions of the new coordinates, defined by the equalities

$$ds_1 = H_1 dq_1; \quad ds_2 = H_2 dq_2, \qquad (2)$$

where ds_1 and ds_2 are the differentials of the arc for the adiabatic curve and the isotherm, respectively.

According to the definition of the new coordinate system

$$\frac{\partial t}{\partial q_1} = 1; \quad \frac{\partial t}{\partial q_2} = 0.$$

Then, it follows from (1) that

$$\frac{\partial}{\partial q_1}\left(\frac{H_2}{H_1}\right) = 0.$$

By exchanging the locations of the isotherms and the adiabatic curves in the specified figure (this operation will subsequently be known as transposition), we derive a figure for which the stream function ψ satisfies the Laplace equation. On the basis of considerations analogous to the above, for a transposed figure we derive

$$\frac{\partial}{\partial q_2}\left(\frac{H_1}{H_2}\right) = 0.$$

It follows from the last two equations that the functions H_1 and H_2 can be presented in the form

$$H_1 = mf(q_1, q_2); \quad H_2 = nf(q_1, q_2), \tag{3}$$

where m and n are constants and f is a function.

Using the coordinate system (q_1, q_2), we find an expression for the conductivity Π of the given figure. By means of two adiabatic curves in proximity to each other we isolate a strip connecting the boundary isotherms. It is not difficult to demonstrate that the conductivity of this strip—with consideration of (2)—is equal to

$$d\Pi = \left[\int\limits_{q_1'}^{q_1''} \frac{ds_1}{ds_2}\right]^{-1} = dq_2\left[\int\limits_{q_1'}^{q_1''} \frac{H_1}{H_2}\, dq_1\right]^{-1},$$

where q_1' and q_1'' are the values of the coordinate q_1 on the boundary isotherms. The conductivity of the entire figure is thus

$$\Pi = \int\limits_{q_2'}^{q_2''} \frac{dq_2}{\int\limits_{q_1'}^{q_1''} \frac{H_1}{H_2}\, dq_1}.$$

Here q_2' and q_2'' are the values of the coordinate q_2 on the boundary adiabatic curves. Having substituted H_1 and H_2 into this formula according to Eqs. (3), we obtain

$$\Pi = \frac{n(q_2'' - q_2')}{m(q_1'' - q_1')}. \tag{4}$$

This result corresponds to the fact that the specified two-dimensional figure in the new coordinates is transformed into a rectangle whose sides—given an appropriate choice of scale for the variables—are equal to $q_2'' - q_2'$ (for the isotherms) and $q_1'' - q_1'$ (for the adiabatic curves).

The area of this figure in coordinates (q_1, q_2) is expressed by the formula

$$S = \int\limits_{q_1'}^{q_1''}\int\limits_{q_2'}^{q_2''} ds_1 ds_2 = mn \int\limits_{q_1'}^{q_1''}\int\limits_{q_2'}^{q_2''} f^2(q_1, q_2)\, dq_1\, dq_2. \tag{5}$$

The preliminary remarks which we have made make it possible to demonstrate the following fundamental statement. If a plane figure is subdivided by an arbitrary adiabatic line which does not coincide with the adiabatic curve of the given figure, the conductivity of the new composite figure will be less than that of the original figure. Since any plane figure in corresponding coordinates represents a rectangle, it is sufficient to prove this statement for the case in which the original figure is a rectangle and the boundary of separation is not a segment of the straight line perpendicular to the isothermal sides of the rectangle (Fig. 1a).

In analogy with the foregoing, for figures I and II we will introduce a new coordinate system (q_1, q_2) in which the figures will be in the form of rectangles. The origin and scale of the independent variables q_1 and q_2 can always be chosen so that these rectangles are situated as shown in Fig. 1b. We will also assume that

$$q_1'' - q_1' \quad \delta. \tag{6}$$

Generally speaking, for figures I and II the expressions for H_1 and H_2 will be different. For figure I, let these expressions have the form of (3), while for figure II

$$H_1^{\mathrm{II}} = MF(q_1, q_2); \quad H_2^{\mathrm{II}} = NF(q_1, q_2). \tag{7}$$

The total conductivity of figures I and II according to formula (4), with consideration of (7) and (6),

$$\Pi_s = \Pi^{\mathrm{I}} + \Pi^{\mathrm{II}} = \frac{n}{m\delta}(q_2^s - q_2') + \frac{N}{M\delta}(q_2'' - q_2^s).$$

It must therefore be demonstrated that the conductivity of the original rectangle l/δ is greater than Π_s, i.e.,

$$l > \frac{n}{m}(q_2^s - q_2') + \frac{N}{M}(q_2'' - q_2^s). \tag{8}$$

Having derived the expression for the total area of the figures I and II in the new coordinates from formula (5) with consideration of (7) and having transformed the corresponding double integrals by using the mean-value theorem, we will have

$$l\delta = mn(q_2^s - q_2') \int\limits_{q_1'}^{q_1''} f^2(q_1, q_2^{\mathrm{I}})\, dq_1 +$$

$$+ MN(q_2'' - q_2^s) \int\limits_{q_1'}^{q_1''} F^2(q_1, q_2^{\mathrm{II}})\, dq_1,$$

where

$$q_2' < q_2^{\mathrm{I}} < q_2^s; \quad q_2^s < q_2^{\mathrm{II}} < q_2''.$$

Let us use the Bunyakovskii inequality [5] for the integral of the square of the function; the last expression is then transformed into the inequality

$$l\delta > \frac{mn(q_2^s - q_2')}{q_1'' - q_1'}\left[\int\limits_{q_1'}^{q_1''} f(q_1, q_2^{\mathrm{I}})\, dq_1\right]^2 +$$

$$+ \frac{MN(q_2'' - q_2^s)}{q_1'' - q_1'}\left[\int\limits_{q_1'}^{q_1''} F(q_1, q_2^{\mathrm{II}})\, dq_1\right]^2. \tag{9}$$

It is not difficult to prove that the brackets contain quantities proportional to the length of certain of the mean adiabatic curves in the corresponding figures. For example, for figure I the length of the adiabatic curve is equal to

$$\int\limits_{q_1'}^{q_1''} ds_1 = \int\limits_{q_1'}^{q_1''} H_1 dq_1 = m \int\limits_{q_1'}^{q_1''} f(q_1, q_2)\, dq_1.$$

It follows directly from Fig. 1a that the smallest length equal to δ is exhibited by the adiabatic curves when $q_2 = q_2'$ and $q_2 = q_2''$. Replacing the integrals in inequality (9) by smaller quantities δ/m and δ/M, canceling like terms, and considering condition (6), we derive inequality (8) which may consequently be regarded as proved.

The resistance R^* of the transposed figure may therefore be demonstrated by means of formula (4) to be equal to the conductivity of the original figure, i.e.,

$$\Pi = R^* = \frac{1}{\Pi^*}. \qquad (10)$$

Plotting the adiabatic boundaries of separation on the plane figure under consideration and calculating the conductivity of the composite figure, we thus obtain the lower bound for Π. Analogous operations for the transposed figure—with relation (10)—yield the upper estimate for Π. Appropriate selection of the division method results in the convergence of these bounds to a point such that the determination of Π by any of the methods is sufficiently accurate for practical purposes. In this case, if a wall of insulation is being calculated for industrial applications, the upper bound should be employed; however, if the wall is intended for conduction, the lower bound should be used to achieve a solution "with a margin of error."

By means of the division method, let us find the approximate formulas to calculate the conductivities of several plane figures.

For a corrugated wall (see, for example, [6]), a parallelogram may serve as the simplest element of the wall. We will denote the sides of this parallelogram by a (isotherms) and b (adiabatic curves), with the acute angle denoted by α.

It follows from (10) that if the figure is not altered in the transposition, its conductivity $\Pi = 1$. Any parallelogram when $a = b$, i.e., a rhombus, is therefore a unit figure. When $a/b = 0$, obviously $\Pi = 0$. We will consider only the parallelograms in the interval $0 < a/b < 1$. When $a/b > 1$, the conductivity can be found from (10).

Let us divide the given figure into narrow strips by means of a system of straight lines, parallel to the boundary adiabatic curves, and let us replace each strip by a rectangle whose sides are equal to $\sin\alpha da$ and $b + \cos\alpha da$, and for which the conductivity is less than the conductivity of the strip and equal to $d\Pi = \sin\alpha da\,(b + \cos\alpha da)^{-1}$. As $da \to 0$, $d\Pi \approx \sin\alpha(da/b)$. The conductivity of the composite figure derived by integration of $d\Pi$ in limits from 0 to a, thus yields the lower estimate for the conductivity of the parallelogram

$$\Pi > \frac{a}{b}\sin\alpha. \qquad (11)$$

The transposed figure will be a parallelogram having boundary isotherms b and adiabatic curves a, with $b > a$. Let us consider the parallelograms with the integral ratio $b/a = i$, and which can be divided by means of the adiabatic lines into i parallel included rhombi. According to the fundamental statement of the method, for such figures $\Pi^* > b/a$. Assuming that Π^* as a function of b/a is monotonic, we can make the statement that for any $b/a > 1$, $\Pi^* > b/a$. The upper estimate for the parallelograms under consideration is therefore

$$\Pi = \frac{1}{\Pi^*} < \frac{a}{b}. \qquad (12)$$

The maximum divergence between estimates (11) and (12) when $a/b = 1$ and $\alpha = \pi/4$ does not exceed 15%; for heat-engineering calculations this is acceptable. However, it is possible to achieve a much more exact solution of the problem by deriving an interpolational formula on the basis of the method described below. We know that one is the exact value of Π for the parallelogram when $a = b$. Let us find the rectangle with the same distance between the boundary adiabatic curves and exhibiting the same conductivity as the rhombus. This will obviously be a square whose sides are equal to $a\sin\alpha$. Having established the length of the boundary isotherms for the two figures and having also established α, we will increase the length of the boundary adiabatic curves, retaining a specific relationship between these. We will refer to these two figures as connected. The form of the connection, i.e., the form of the relationship between the length of the boundary adiabatic curves, must be chosen so that the conductivity of the connected figures coincides at least for one value of a geometric parameter. If the exact formula for Π is known for one of the figures, it can serve as an approximate formula for the Π of another figure which is more complex.

For the connected parallelogram and rectangle we will assume that the difference between the length of the boundary adiabatic curves remains constant. The length of the adiabatic curve for the rectangle is thus equal to $b - a(1 - \sin\alpha)$ and for $a/b \le 1$ the conductivity of the parallelogram is

$$\Pi \approx \frac{a\sin\alpha}{b - a(1 - \sin\alpha)} = \frac{\sin\alpha}{\frac{b}{a} + \sin\alpha - 1}. \qquad (13)$$

It is not difficult to prove that according to formula (13) the value of Π does not exceed the limits of estimates (11) and (12), while for $a/b = 0$ formula (13) yields an exact value of $\Pi = 0$. Control calculations for the parallelogram conductivities with intermediate values of a/b by means of a network method demonstrated the high accuracy of formula (13) over the entire interval $0 \le a/b \le 1$. For example, when $a/b = 0.5$ and $\alpha = \pi/4$, the Π calculated from formula (13) and the magnitude of Π derived numerically for a network grid of b/14, differ by less than 3%.

The problem of the conductivity in a closed shell bounded in the lateral cross section from within and without by concentric regular polygons calls for consideration of a rectangular trapezoid. We will denote the bases (isotherms) of the trapezoid as a and b ($a < b$), while the acute angle formed by the boundary adiabatic curves is denoted by α. The method of dividing this figure becomes clear from Fig. 2a. Replacing each strip by a sector of the concentric ring and calculating the conductivity of the composite figure, we derive the lower estimate for the conductivity of the trapezoid:

$$\Pi > \int_0^{\alpha} \frac{d\varphi}{\ln \frac{b}{a}} = \frac{\alpha}{\ln \frac{b}{a}}. \tag{14}$$

The division of the transposed figure is shown in Fig. 2b. Here we obtain strips of constant width dx, as dx → 0 exhibiting a length $a + \alpha x$. The estimate for the conductivity of the transposed figure is

$$\Pi^* > \int_0^{(b-a)\,\mathrm{ctg}\,\alpha} \frac{dx}{a + \alpha x} =$$

$$= \frac{1}{\alpha} \ln \left[1 + \alpha \left(\frac{b}{a} - 1 \right) \mathrm{ctg}\,\alpha \right].$$

Consequently, the conductivity of the given figure is bounded from above by the inequality

$$\Pi < \frac{\alpha}{\ln \left[1 + \alpha \left(\frac{b}{a} - 1 \right) \mathrm{ctg}\,\alpha \right]}. \tag{15}$$

The maximum divergence in the estimates for $\alpha = \pi/4$ is attained when $b/a = 1.1$ and amounts to 20%. The quantity a/b may vary from 0 to 1. Each of the estimates yields identical values for Π at the ends of this interval. For the derivation of the interpolational formula we have to know at least one other value of Π within the interval. For this purpose, let us consider

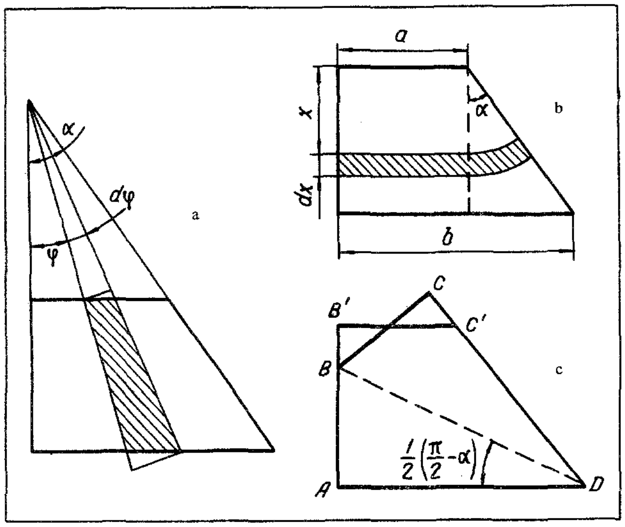

Fig. 2. Approximate determination of conductivity of rectangular trapezoid: a) division of given trapezoid, b) division of transposed trapezoid; c) trapezoid determination with $\Pi \approx 1$.

the figure ABCD (Fig. 2c) in which the sides AB and CD are adiabatic curves, BC and AD are isotherms, with AD = CD = b and AB = BC = a, and the angle between the adiabatic curves denoted by α. It follows from the above that for such a figure $\Pi = 1$. Let us complete the transition from the figure ABCD to a trapezoid of the same area of the same isotherm length (B'C' = BC), and the same average length for the boundary adiabatic curves (BB' = CC'). With this transition,

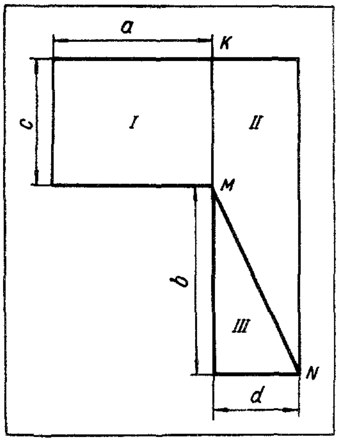

Fig. 3. Wall element with conductors of rectangular section.

the conductivity undergoes no substantial change. It is therefore possible to assume that for all trapezoids with $a/b = \mathrm{tg}((\pi/4) - (\alpha/2))$, $\Pi \approx 1$. As a figure connected with the trapezoid, let us take the sector of the concentric ring with the central angle α, and as the form of the connection we will take the function $(b/a)/(r_2/r_1) = \mathrm{const}$, where r_2/r_1 is the ratio of the ring radii. For the sector $\Pi = 1$, $r_2/r_1 = \exp \alpha$. The approximate formula for the conductivity of the trapezoid will then have the form

$$\Pi \approx \frac{\alpha}{\ln \left[\frac{b}{a}\, \mathrm{tg} \left(\frac{\pi}{4} - \frac{\alpha}{2} \right) \right] + \alpha}. \tag{16}$$

Formula (16) yields excellent results for $a/b \le \mathrm{tg}((\pi/4) - (\alpha/2))$, but with a/b close to 1, the values of Π calculated from this formula exceed the limits of estimate (15). For the connected figure it is therefore better to select a rectangle in which the length of the adiabatic curve is equal to the height h of the trapezoid. Having established h and α, we will increase the lengths of the boundary isotherms of the connected figures, retaining a constant difference. As a result, we derive the following interpolational formula for $a/b \ge \mathrm{tg}((a/4 - (\alpha/2))$:

$$\Pi \approx 1 + \mathrm{tg}\,\alpha \left[\frac{1}{\frac{b}{a} - 1} - \frac{1}{\mathrm{ctg}\left(\frac{\pi}{4} - \frac{\alpha}{2} \right) - 1} \right]. \tag{17}$$

Formulas (16) and (17) have been checked for various values of α and a/b by determining the value of Π through a numerical method. The agreement of the results is excellent. Thus, when $\alpha = \pi/4$ and $a/b = 0.5$, the divergence in the magnitude of Π amounts to ~1%, while for $a/b = 0.1$, it is approximately 4% (the network spacing was taken as equal to 0.1b).

Let us point to yet another method of deriving the approximate formula for the conductivity of complex figures. We will examine this method by calculating the insulation within which rows of parallel rectangular conductors are positioned uniformly [3]. The problem reduces to the determination of the conductivity of the two-dimensional figure shown in Fig. 3. We will divide this figure into three parts by means of the isothermal line MN and with the adiabatic line KM. It follows from the above that the first operation increases conductivity, while the second operation reduces the conductivity of the derived composite figure, relative to the original figure. However, since the lines MN and KM are close to the isotherm and to the adiabatic curve of the given figure (we can prove this by using the familiar method of constructing isotherms and adiabatic curves [6]), we should expect the conductivity of the original figure to be approximately equal to the conductivity of the original figure to be approximately equal to the conductivity of the composite figure:

$$\Pi \approx \Pi^{\mathrm{I}} + (R^{\mathrm{II}} + R^{\mathrm{III}})^{-1} = \Pi^{\mathrm{I}} + \Pi^{\mathrm{II}}.$$

Here $R^{\mathrm{III}} = 0$, since the boundary isotherms in figure III touch. Figure I is a rectangle and figure II is a transposed trapezoid. Using Eqs. (10), (16), and (17), with the denotations of Fig. 3, we derive

$$\Pi \approx \frac{a}{c} + \left(\operatorname{arctg}\frac{b}{d}\right)^{-1} \times$$

$$\times \ln\left[\left(\frac{b}{c}+1\right)\frac{d}{b+\sqrt{b^2+d^2}}\right]+1 \qquad (18)$$

for

$$\frac{b}{c} \geqslant \frac{b+\sqrt{b^2+d^2}}{d} - 1;$$

$$\Pi \approx \frac{a}{c} + \left(1 + \frac{c}{d} - \frac{b}{b-d+\sqrt{b^2+d^2}}\right)^{-1} \qquad (19)$$

for

$$\frac{b}{c} \leqslant \frac{b+\sqrt{b^2+d^2}}{d} - 1.$$

Formulas (18) and (19) have been checked by a numerical method. For small values of d/b we find the results to be in excellent agreement. For example, when $a = 3$, b = 6, c = 4, and d = 2 the calculation by a numerical method for a network spacing of 0.5d yields $\Pi = 1.170$, whereas according to formula (19) we have $\Pi = 1.163$.

NOTATION

Q is the heat flux; λ is the thermal conductivity; t is the temperature; ψ is the stream function; q_1 and q_2 are the orthogonal curvilinear coordinates which coincide with the system of isothermal and adiabatic curves of the plane figure; H_1 and H_2 are functions of the coordinates q_1 and q_2; ds_1 and ds_2 are the differentials of the adiabatic and isothermal arcs; m, n, M, and N are constants; f and F are symbols of functions; Π is the conductivity of the plane figure; R is the resistance of the plane figure; Π^* and R^* are the conductivity and resistance of the transposed figure; S is the figure area; l, δ, a, b, c, d, and h are various linear dimensions; r_2 and r_1 are the large and small radii of the ring; α and φ are angles; i is an integer; x is a coordinate.

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